

공학수학 1

2009년도 1학기

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# 1.  $x y''' + 3 y'' = x \cdot e^x$  의 특수해를 Wronskian 을 이용하여 구하시오.

비제차 3차 방정식의 형태이므로,

보조방정식  $\lambda(\lambda-1)(\lambda-2) + 3\lambda(\lambda-1) = \lambda^3 - \lambda = 0$  으로부터,

$y_1 = 1$ ,  $y_2 = x$ ,  $y_3 = \frac{1}{x}$  의 제차해를 얻는다.

미분방정식의 표준형은  $y''' + \frac{3}{x} y'' = e^x$  이므로, 비제차항  $h(x) = e^x$ .

$$W = \begin{vmatrix} 1 & x & x^{-1} \\ 0 & 1 & -x^{-2} \\ 0 & 0 & 2x^{-3} \end{vmatrix} = \frac{2}{x^3}$$

$$w_1 = \begin{vmatrix} 0 & x & x^{-1} \\ 0 & 1 & -x^{-2} \\ e^x & 0 & 2x^{-3} \end{vmatrix} = -\frac{2e^x}{x}, \quad w_2 = \begin{vmatrix} 1 & 0 & x^{-1} \\ 0 & 0 & -x^{-2} \\ 0 & e^x & 2x^{-3} \end{vmatrix} = \frac{e^x}{x^2}, \quad w_3 = \begin{vmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^x \end{vmatrix} = e^x$$

$$y_p = y_1 \int \frac{w_1}{W} + y_2 \int \frac{w_2}{W} + y_3 \int \frac{w_3}{W} \quad \text{이므로,}$$

$$\begin{aligned} \therefore y_p &= 1 \cdot \int \frac{x^3}{2} \cdot \left(-\frac{2e^x}{x}\right) dx + x \cdot \int \frac{x^3}{2} \cdot \left(\frac{e^x}{x^2}\right) dx + \frac{1}{x} \int \frac{x^3}{2} \cdot (e^x) dx \\ &= - \int x^2 \cdot e^x dx + \frac{x}{2} \int x \cdot e^x dx + \frac{1}{2x} \int x^3 \cdot e^x dx \\ &= -(x^2 - 2x + 2) e^x + \frac{x}{2} (x-1) e^x + \frac{1}{2x} (x^3 - 3x^2 + 6x - 6) \cdot e^x \\ &= \left(1 - \frac{3}{x}\right) e^x \end{aligned}$$



# 2. Frobenius method 을 이용하여, 미분방정식의 해의 기저계를 구하시오.

$$2x(x-1)y'' - (x+1)y' + y = 0$$

$y = x^r \cdot \sum_{m=0}^{\infty} a_m \cdot x^m$  을 원식에 대입하면,

$$2 \sum_{m=0}^{\infty} (m+r)(m+r-1) \cdot a_m x^{m+r} - 2 \sum_{m=-1}^{\infty} (m+r+1)(m+r) a_{m+1} x^{m+r} \\ - \sum_{m=0}^{\infty} (m+r) a_m \cdot x^{m+r} - \sum_{m=-1}^{\infty} (m+r+1) a_{m+1} \cdot x^{m+r} + \sum_{m=0}^{\infty} a_m \cdot x^{m+r} = 0.$$

$$m=-1 \text{ 일 때, } -2r(r-1)a_0 - r a_0 = 0$$

$$\therefore r = 0 \text{ or } \frac{1}{2}$$

①  $r = 0$  인 경우,

$$2m(m-1)a_m - 2(m+1)m a_{m+1} - m a_m - (m+1)a_{m+1} + a_m \\ = -(m+1)(2m+1)a_{m+1} + (2m^2 - 3m + 1)a_m = 0$$

$$\therefore a_{m+1} = \frac{(2m-1)(m-1)}{(m+1)(2m+1)} \cdot a_m \quad (m \geq 0)$$

$$\text{위식에서 } a_0 = a_1, a_2 = a_3 = \dots = 0 \text{ 이므로, } y_1 = 1 + x$$

②  $r = \frac{1}{2}$  인 경우,

$$2\left(m+\frac{1}{2}\right)\left(m-\frac{1}{2}\right)a_m - 2\left(m+\frac{3}{2}\right)\left(m+\frac{1}{2}\right)a_{m+1} - \left(m+\frac{1}{2}\right)a_m - \left(m+\frac{3}{2}\right)a_{m+1} + a_m = 0.$$

$$\therefore \left(m+\frac{3}{2}\right)(2m+2)a_{m+1} = (2m^2 - m)a_m \quad (m \geq 0)$$

$$\text{위식에서 } a_1 = a_2 = a_3 = \dots = 0. \text{ 이므로, } y_2 = \sqrt{x}$$



# 3.

$$x^2 y'' + \frac{1}{4} \left( x + \frac{10^2 - 2^2}{10^2} \right) y = 0$$

$$y = u \sqrt{x}, \quad \sqrt{x} = z \quad \text{이므로}$$

$$y' = u' \sqrt{x} + \frac{1}{2} x^{-\frac{1}{2}} u = x^{\frac{1}{2}} \cdot \frac{dz}{dx} \cdot \frac{du}{dz} + \frac{1}{2} x^{-\frac{1}{2}} u$$

$$= x^{\frac{1}{2}} \left( \frac{1}{2} x^{-\frac{1}{2}} \right) \dot{u} + \frac{1}{2} x^{-\frac{1}{2}} u$$

$$= \frac{1}{2} \dot{u} + \frac{1}{2z} u$$

$$y'' = \left( \frac{1}{2} \ddot{u} + \frac{1}{2z} \dot{u} - \frac{1}{2z^2} u \right) \frac{dz}{dx} = \frac{1}{4z} \ddot{u} + \frac{1}{4z^2} \dot{u} - \frac{1}{4z^3} u$$

위에서 구한  $y'$  와  $y''$  를 준식에 대입하면,

$$\frac{z^3}{4} \ddot{u} + \frac{z^2}{4} \dot{u} - \frac{z}{4} u + \frac{1}{4} \left( z^2 + \frac{10^2 - 2^2}{10^2} \right) z u = 0$$

$$z^2 \ddot{u} + z \dot{u} + \left( z^2 - \frac{2^2}{10^2} \right) u = 0$$

$$\therefore u = C_1 J_{\frac{2}{10}}(z) + C_2 J_{-\frac{2}{10}}(z)$$

$$\therefore y = \sqrt{x} \left[ C_1 J_{\frac{2}{10}}(z) + C_2 J_{-\frac{2}{10}}(z) \right]$$



# 4.

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} \cdot J_{\nu}(x) \quad \dots \text{공식 1.}$$

$$\nu = -\frac{1}{2} \text{ 을 대입하면, } J_{-\frac{3}{2}}(x) = -J_{\frac{1}{2}}(x) - \frac{1}{x} \cdot J_{\frac{1}{2}}(x)$$

$$\text{문제에서, } J_{\frac{1}{2}} = \sqrt{\frac{2}{\pi x}} \cdot \sin x, \quad J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}} \cdot \cos x \quad \text{이므로, 이를 위변에 대입하면,}$$

$$\therefore J_{-\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}} \left( \frac{\cos x}{x} + \sin x \right)$$

여기서, 공식 1의 증명은,

$$\begin{array}{lcl} (x^{\nu} \cdot J_{\nu})' = x^{\nu} \cdot J_{\nu-1} & \xrightarrow{\text{전개}} & \nu \cdot x^{\nu-1} \cdot J_{\nu} + x^{\nu} \cdot J_{\nu}' = x^{\nu} \cdot J_{\nu-1} \\ (x^{-\nu} \cdot J_{\nu})' = -x^{-\nu} \cdot J_{\nu+1} & & -\nu \cdot x^{-\nu-1} \cdot J_{\nu} + x^{-\nu} \cdot J_{\nu}' = -x^{-\nu} \cdot J_{\nu+1} \end{array}$$

위의 첫식에는  $x^{\nu}$ 를, 두번째 식에는  $x^{-\nu}$ 를 곱하여 정리하면

$$2 \dots \frac{d}{dx} J_{\nu}(x) = J_{\nu-1}(x) - \frac{\nu}{x} \cdot J_{\nu}(x)$$

$$3 \dots \frac{d}{dx} J_{\nu}(x) = -J_{\nu+1}(x) + \frac{\nu}{x} \cdot J_{\nu}(x) \quad \text{을 연고, 식 2에서 3을 빼면, 공식 1을 얻는다.}$$

# 5. Laplace Transform / Inverse transform

$$(1) \sinh(2t) \cdot \cos t$$

$$\mathcal{L}(\cos t) = \frac{s}{s^2+1} \quad \text{이므로,}$$

$$\mathcal{L}\{\sinh(2t) \cdot \cos t\} = \mathcal{L}\left(\frac{e^{2t} - e^{-2t}}{2} \cdot \cos t\right) = \mathcal{L}\left(\frac{1}{2}e^{2t} \cdot \cos t - \frac{1}{2}e^{-2t} \cdot \cos t\right)$$

$$= \frac{1}{2} \cdot \frac{(s-2)}{(s-2)^2+1} - \frac{1}{2} \cdot \frac{(s+2)}{(s+2)^2+1}$$

$$= \frac{2s^2-10}{s^4-6s^2+25}$$



$$(2) (e^{2-t} - t) \cdot U(t-2)$$

$$\begin{aligned} \mathcal{L}(e^{2-t} \cdot U(t-2)) &= \int_0^{\infty} e^{-st} \cdot e^{2-t} \cdot U(t-2) dt \\ &= e^2 \int_0^2 e^{-(s+1)t} dt + e^2 \int_2^{\infty} e^{-(s+1)t} dt \\ &= e^2 \left[ -\frac{1}{s+1} \cdot e^{-(s+1)t} \right]_2^{\infty} = \frac{1}{s+1} \cdot e^{-2s} \end{aligned}$$

$$\begin{aligned} \mathcal{L}(-t \cdot U(t-2)) &= - \int_0^{\infty} e^{-st} \cdot t \cdot U(t-2) dt \\ &= - \int_2^{\infty} e^{-st} \cdot t dt = \left[ \frac{1}{s} \cdot e^{-st} \cdot t \right]_2^{\infty} - \frac{1}{s} \int_2^{\infty} e^{-st} dt \\ &= -\frac{2}{s} \cdot e^{-2s} - \frac{1}{s^2} \cdot e^{-2s} \\ &= -\left(\frac{2}{s} + \frac{1}{s^2}\right) \cdot e^{-2s} \end{aligned}$$

$$\therefore \mathcal{L}\{(e^{2-t} - t) \cdot U(t-2)\} = \left(\frac{1}{s+1} - \frac{2}{s} - \frac{1}{s^2}\right) \cdot e^{-2s}$$

$$(3) \frac{1}{s(s^2+4)}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2} \cdot \sin 2t \quad 0 \leq t < \infty,$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \frac{1}{2} \int_0^t \sin 2\tau d\tau = \left[-\frac{1}{4} \cos 2\tau\right]_0^t = \frac{1}{4}(1 - \cos 2t)$$

$$(4) \frac{s \cdot e^{-4s}}{s^2 + \pi^2}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2 + \pi^2}\right\} = \cos \pi t \quad 0 \leq t < \infty,$$

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s \cdot e^{-4s}}{s^2 + \pi^2}\right\} &= \cos \pi(t-4) \cdot U(t-4) \\ &= \cos \pi t \cdot U(t-4) \end{aligned}$$



# 6. Laplace transform

$$y'' - 5y' + 6y = \begin{cases} 4e^t & (0 < t < 2) \\ 0 & (t > 2) \end{cases}, \quad y(0) = 1, \quad y'(0) = 2$$

$$\textcircled{1} \quad r(t) = 4e^t \{1 - u(t-2)\} = 4e^t - 4e^2 \cdot e^{t-2} \cdot u(t-2)$$

$$\mathcal{L}(r(t)) = \frac{4}{s-1} - 4e^2 \frac{e^{-2s}}{s-1}$$

$$\textcircled{2} \quad \mathcal{L}(y'') = s^2 \mathcal{L}(y) - s \cdot y(0) - y'(0) = s^2 \mathcal{L}(y) - s + 2$$

$$\mathcal{L}(y') = s \mathcal{L}(y) - y(0) = s \mathcal{L}(y) - 1.$$

$\textcircled{1}$  과  $\textcircled{2}$  의 결과를 이용, 양변을 라플라스 변환하면,

$$(s^2 - 5s + 6) \cdot \mathcal{L}(y) = s - 1 + \frac{4}{s-1} - 4e^2 \cdot \frac{e^{-2s}}{s-1}$$

$$\therefore \mathcal{L}(y) = \frac{s^2 - 8s + 11 - 4e^2 \cdot e^{-2s}}{(s-1)(s-2)(s-3)}$$

$$= \frac{2}{s-1} + \frac{1}{s-2} - \frac{2}{s-3} - \frac{4e^2 \cdot e^{-2s}}{(s-1)(s-2)(s-3)}$$

따라서,  $0 < t < 2$  일때,

$$\begin{cases} y = 2e^t + e^{2t} - 2e^{3t} \\ t > 2 \text{ 일때,} \end{cases}$$

$$y = (1 + 4e^2)e^{2t} - (2 + 2e^4)e^{3t}$$



## # 17. Gauss 소대법

$$\begin{aligned}x_1 - 2x_2 - x_3 + 3x_4 &= -3 \\ -3x_1 + 7x_2 + 3x_3 - 11x_4 &= 13 \\ 4x_1 - 10x_2 - x_3 + 10x_4 &= -26\end{aligned}$$

주어진 연립방정식을 첨가행렬을 이용하여 나타내면,

$$\begin{pmatrix} 1 & -2 & -1 & 3 & -3 \\ -3 & 7 & 3 & -11 & 13 \\ 4 & -10 & -1 & 10 & -26 \end{pmatrix}$$

1행에 3을 곱해서 2행에 더하고, 1행에 4를 곱해서 3행에 더하면,

$$\begin{pmatrix} 1 & -2 & -1 & 3 & -3 \\ 0 & 1 & 0 & -2 & 4 \\ 0 & -2 & 3 & -2 & -14 \end{pmatrix}$$

이후, 2행에 2를 곱해서, 3행에 더하면,

$$\begin{pmatrix} 1 & -2 & -1 & 3 & -3 \\ 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 3 & -6 & -6 \end{pmatrix}$$

위의 식에  $x_4 = t$  로 두고, 후진대입 (Backward Substitution) 에 의해,

$$x_4 = t.$$

$$x_3 = 2x_4 - 2 = 2t - 2$$

$$x_2 = 2x_4 + 4 = 2t + 4$$

$$x_1 = 2x_2 + x_3 - 3x_4 - 3 = 3t + 3$$



# 8. 
$$\begin{pmatrix} 5 & 10 & -10 \\ 10 & 5 & -20 \\ 5 & -5 & -10 \end{pmatrix}$$

①  $\det(A - \lambda I) = -\lambda^3 + 225\lambda = 0$

$\therefore \lambda_1 = 15, \lambda_2 = 0, \lambda_3 = -15$

$A - 15I = \begin{bmatrix} -10 & 10 & -10 \\ 10 & -10 & -20 \\ 5 & -5 & -25 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \chi_1 = [1 \ 1 \ 0]^T$

$A = \begin{bmatrix} 5 & 10 & -10 \\ 10 & 5 & -20 \\ 5 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \chi_2 = [2 \ 0 \ 1]^T$

$A + 15I = \begin{bmatrix} 20 & 10 & -10 \\ 10 & 20 & -20 \\ 5 & -5 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \chi_3 = [0 \ 1 \ 1]^T$

$\therefore X = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, X^{-1} = -\frac{1}{3} \begin{bmatrix} -1 & -2 & 2 \\ -1 & 1 & -1 \\ 1 & -1 & -2 \end{bmatrix}$

$D = X^{-1}AX = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -15 \end{bmatrix}$



### # 9. 2차형식

$Q = 3x_1^2 - 4x_1x_2 + 3x_2^2 = 5$  를 주축형으로 바꾸어 어떤 원뿔곡선인가 설명하고, 회전각을 구하시오.

$$Q = x^T A x \text{ 에서,}$$

$$A = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ 이고,}$$

$$\text{특성방정식 } (3-\lambda)^2 - 4 = 0 \text{ 이다. } \lambda_1 = 1, \lambda_2 = 5.$$

$$Q = y^T D y = y_1^2 + 5y_2^2 = 5.$$

즉,  $y_1, y_2$  좌표계에서는 "타원"이 된다.

장축의 길이는  $2\sqrt{5}$ , 단축의 길이는 2.

좌표의 변환관계를 구하기 위해, 고유벡터를  $(A - \lambda I)x = 0$  으로부터, 구하면

$$x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

이를 단위 고유벡터로 나타내면,  $X = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$  이고, 즉,  $45^\circ$ 의 회전을 의미한다.